Computer Graphics 4: Bresenham Line Drawing Algorithm, Circle Drawing & Polygon Filling
In today’s lecture we’ll have a look at:

– Bresenham’s line drawing algorithm
– Line drawing algorithm comparisons
– Circle drawing algorithms
  • A simple technique
  • The mid-point circle algorithm
– Polygon fill algorithms
– Summary of raster drawing algorithms
The Bresenham algorithm is another incremental scan conversion algorithm. The big advantage of this algorithm is that it uses only integer calculations.
Move across the $x$ axis in unit intervals and at each step choose between two different $y$ coordinates.

For example, from position $(2, 3)$ we have to choose between $(3, 3)$ and $(3, 4)$. We would like the point that is closer to the original line.
At sample position $x_k + 1$ the vertical separations from the mathematical line are labelled $d_{upper}$ and $d_{lower}$.

The $y$ coordinate on the mathematical line at $x_k + 1$ is:

$$y = m(x_k + 1) + b$$
So, $d_{upper}$ and $d_{lower}$ are given as follows:

$$d_{lower} = y - y_k$$
$$= m(x_k + 1) + b - y_k$$

and:

$$d_{upper} = (y_k + 1) - y$$
$$= y_k + 1 - m(x_k + 1) - b$$

We can use these to make a simple decision about which pixel is closer to the mathematical line.
This simple decision is based on the difference between the two pixel positions:

\[ d_{\text{lower}} - d_{\text{upper}} = 2m(x_k + 1) - 2y_k + 2b - 1 \]

Let’s substitute \( m \) with \( \Delta y/\Delta x \) where \( \Delta x \) and \( \Delta y \) are the differences between the end-points:

\[
\Delta x(d_{\text{lower}} - d_{\text{upper}}) = \Delta x \left(2 \frac{\Delta y}{\Delta x} (x_k + 1) - 2y_k + 2b - 1\right)
\]

\[
= 2\Delta y \cdot x_k - 2\Delta x \cdot y_k + 2\Delta y + \Delta x(2b - 1)
\]

\[
= 2\Delta y \cdot x_k - 2\Delta x \cdot y_k + c
\]
So, a decision parameter $p_k$ for the $k$th step along a line is given by:

$$p_k = \Delta x (d_{\text{lower}} - d_{\text{upper}})$$

$$= 2\Delta y \cdot x_k - 2\Delta x \cdot y_k + c$$

The sign of the decision parameter $p_k$ is the same as that of $d_{\text{lower}} - d_{\text{upper}}$. If $p_k$ is negative, then we choose the lower pixel, otherwise we choose the upper pixel.
Remember coordinate changes occur along the $x$ axis in unit steps so we can do everything with integer calculations.

At step $k+1$ the decision parameter is given as:

$$p_{k+1} = 2\Delta y \cdot x_{k+1} - 2\Delta x \cdot y_{k+1} + c$$

Subtracting $p_k$ from this we get:

$$p_{k+1} - p_k = 2\Delta y (x_{k+1} - x_k) - 2\Delta x (y_{k+1} - y_k)$$
But, \( x_{k+1} \) is the same as \( x_k + 1 \) so:

\[
p_{k+1} = p_k + 2\Delta y - 2\Delta x(y_{k+1} - y_k)
\]

where \( y_{k+1} - y_k \) is either 0 or 1 depending on the sign of \( p_k \).

The first decision parameter \( p_0 \) is evaluated at \((x_0, y_0)\) is given as:

\[
p_0 = 2\Delta y - \Delta x
\]
## BRESENHAM’S LINE DRAWING ALGORITHM
(for $|m| < 1.0$)

1. Input the two line end-points, storing the left end-point in $(x_0, y_0)$

2. Plot the point $(x_0, y_0)$

3. Calculate the constants $\Delta x$, $\Delta y$, $2\Delta y$, and $(2\Delta y - 2\Delta x)$ and get the first value for the decision parameter as:

$$p_0 = 2\Delta y - \Delta x$$

4. At each $x_k$ along the line, starting at $k = 0$, perform the following test. If $p_k < 0$, the next point to plot is $(x_k+1, y_k)$ and:

$$p_{k+1} = p_k + 2\Delta y$$
Otherwise, the next point to plot is \((x_k + 1, y_k + 1)\) and:

\[ p_{k+1} = p_k + 2\Delta y - 2\Delta x \]

5. Repeat step 4 \((\Delta x - 1)\) times

**ACHTUNG!** The algorithm and derivation above assumes slopes are less than 1. For other slopes we need to adjust the algorithm slightly.
Let’s have a go at this
Let’s plot the line from (20, 10) to (30, 18)
First off calculate all of the constants:

- \( \Delta x \): 10
- \( \Delta y \): 8
- \( 2\Delta y \): 16
- \( 2\Delta y - 2\Delta x \): -4

Calculate the initial decision parameter \( p_0 \):

\[- p_0 = 2\Delta y - \Delta x = 6\]
Bresenham Example (cont...)

\[ p_k = \begin{cases} 
1 & \text{if } p_k \geq 0 \\
0 & \text{otherwise} 
\end{cases} \]

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Bresenham Exercise

Go through the steps of the Bresenham line drawing algorithm for a line going from (21,12) to (29,16)
Bresenham Exercise (cont...)

\[ p_k = (x_{k+1}, y_{k+1}) \]

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The Bresenham line algorithm has the following advantages:

- An fast incremental algorithm
- Uses only integer calculations

Comparing this to the DDA algorithm, DDA has the following problems:

- Accumulation of round-off errors can make the pixelated line drift away from what was intended
- The rounding operations and floating point arithmetic involved are time consuming
The equation for a circle is:

\[ x^2 + y^2 = r^2 \]

where \( r \) is the radius of the circle

So, we can write a simple circle drawing algorithm by solving the equation for \( y \) at unit \( x \) intervals using:

\[ y = \pm \sqrt{r^2 - x^2} \]
A Simple Circle Drawing Algorithm (cont…)

\begin{align*}
y_0 &= \sqrt{20^2 - 0^2} \approx 20 \\
y_1 &= \sqrt{20^2 - 1^2} \approx 20 \\
y_2 &= \sqrt{20^2 - 2^2} \approx 20 \\
\vdots \\
y_{19} &= \sqrt{20^2 - 19^2} \approx 6 \\
y_{20} &= \sqrt{20^2 - 20^2} \approx 0
\end{align*}
However, unsurprisingly this is not a brilliant solution!

Firstly, the resulting circle has large gaps where the slope approaches the vertical.

Secondly, the calculations are not very efficient.

- The square (multiply) operations
- The square root operation – try really hard to avoid these!

We need a more efficient, more accurate solution.
The first thing we can notice to make our circle drawing algorithm more efficient is that circles centred at \((0, 0)\) have *eight-way symmetry*. 

![Diagram showing eight-way symmetry of a circle centered at (0, 0).](image)
Similarly to the case with lines, there is an incremental algorithm for drawing circles – the *mid-point circle algorithm*. In the mid-point circle algorithm we use eight-way symmetry so only ever calculate the points for the top right eighth of a circle, and then use symmetry to get the rest of the points.

The mid-point circle algorithm was developed by Jack Bresenham, who we heard about earlier. Bresenham’s patent for the algorithm can be viewed [here](#).
Assume that we have just plotted point \((x_k, y_k)\). The next point is a choice between \((x_k+1, y_k)\) and \((x_k+1, y_k-1)\). We would like to choose the point that is nearest to the actual circle. So how do we make this choice?
Mid-Point Circle Algorithm (cont…)

Let’s re-jig the equation of the circle slightly to give us:

\[ f_{\text{circ}}(x, y) = x^2 + y^2 - r^2 \]

The equation evaluates as follows:

\[
\begin{align*}
&< 0, \text{ if } (x, y) \text{ is inside the circle boundary} \\
&= 0, \text{ if } (x, y) \text{ is on the circle boundary} \\
&> 0, \text{ if } (x, y) \text{ is outside the circle boundary}
\end{align*}
\]

By evaluating this function at the midpoint between the candidate pixels we can make our decision.
Assuming we have just plotted the pixel at $(x_k, y_k)$ so we need to choose between $(x_k + 1, y_k)$ and $(x_k + 1, y_k - 1)$

Our decision variable can be defined as:

$$p_k = f_{circ}(x_k + 1, y_k - \frac{1}{2})$$

$$= (x_k + 1)^2 + (y_k - \frac{1}{2})^2 - r^2$$

If $p_k < 0$ the midpoint is inside the circle and the pixel at $y_k$ is closer to the circle

Otherwise the midpoint is outside and $y_k - 1$ is closer
To ensure things are as efficient as possible we can do all of our calculations incrementally

First consider:

\[ p_{k+1} = f_{\text{circ}} \left( x_{k+1} + 1, y_{k+1} - \frac{1}{2} \right) \]

\[ = [(x_k + 1) + 1]^2 + \left( y_{k+1} - \frac{1}{2} \right)^2 - r^2 \]

or:

\[ p_{k+1} = p_k + 2(x_k + 1) + (y_{k+1} - y_k)^2 - (y_{k+1} - y_k) + 1 \]

where \( y_{k+1} \) is either \( y_k \) or \( y_k - 1 \) depending on the sign of \( p_k \)
The first decision variable is given as:

$$p_0 = f_{circ}(1, r - \frac{1}{2})$$

$$= 1 + (r - \frac{1}{2})^2 - r^2$$

$$= \frac{5}{4} - r$$

Then if $p_k < 0$ then the next decision variable is given as:

$$p_{k+1} = p_k + 2x_{k+1} + 1$$

If $p_k > 0$ then the decision variable is:

$$p_{k+1} = p_k + 2x_{k+1} + 1 - 2y_k + 1$$
The Mid-Point Circle Algorithm

MID-POINT CIRCLE ALGORITHM

• Input radius $r$ and circle centre $(x_c, y_c)$, then set the coordinates for the first point on the circumference of a circle centred on the origin as:

$$(x_0, y_0) = (0, r)$$

• Calculate the initial value of the decision parameter as:

$$p_0 = \frac{5}{4} - r$$

• Starting with $k = 0$ at each position $x_k$, perform the following test. If $p_k < 0$, the next point along the circle centred on $(0, 0)$ is $(x_k+1, y_k)$ and:

$$p_{k+1} = p_k + 2x_{k+1} + 1$$
Otherwise the next point along the circle is \((x_k+1, y_k-1)\) and:

\[ p_{k+1} = p_k + 2x_{k+1} + 1 - 2y_{k+1} \]

4. Determine symmetry points in the other seven octants
5. Move each calculated pixel position \((x, y)\) onto the circular path centred at \((x_c, y_c)\) to plot the coordinate values:

\[ x = x + x_c \quad y = y + y_c \]

6. Repeat steps 3 to 5 until \(x \geq y\)
To see the mid-point circle algorithm in action let's use it to draw a circle centred at (0,0) with radius 10
Use the mid-point circle algorithm to draw the circle centred at (0,0) with radius 15
Mid-Point Circle Algorithm Example (cont…)

\[ p_k = \frac{y_{k+1} - x_k}{2} \]

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The key insights in the mid-point circle algorithm are:

- Eight-way symmetry can hugely reduce the work in drawing a circle
- Moving in unit steps along the x axis at each point along the circle’s edge we need to choose between two possible y coordinates
So we can figure out how to draw lines and circles
How do we go about drawing polygons?
We use an incremental algorithm known as the scan-line algorithm
Scan-Line Polygon Fill Algorithm
The basic scan-line algorithm is as follows:

– Find the intersections of the scan line with all edges of the polygon
– Sort the intersections by increasing x coordinate
– Fill in all pixels between pairs of intersections that lie interior to the polygon
Scan-Line Polygon Fill Algorithm (cont...).
Over the last couple of lectures we have looked at the idea of scan converting lines. The key thing to remember is this has to be **FAST**.

For lines we have either DDA or Bresenham.

For circles the mid-point algorithm.
Anti-Aliasing
Summary Of Drawing Algorithms
Mid-Point Circle Algorithm (cont...)

The image shows a diagram of a grid with a line tracing the points of a circle. The points are labeled with numbers from 1 to 6, and the line connects these points in a smooth curve, representing the circle's edge. The grid is structured in a way that each point is equidistant from its neighbors, forming a symmetrical pattern.
Mid-Point Circle Algorithm (cont...)

[Diagram of a grid with numbered points and a line segment labeled M]